THE EXPRESSION

OF

A QUADRATIC SURD

AS A

CONTINUED FRACTION.

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PREFATORY NOTE.

In investigating the properties of a new special form of determinant intimately connected with the subject of continued fractions,* it was necessary for the author to study somewhat minutely that particular class of continued fractions which are expressible as quadratic surds. The accounts given of this class in the ordinary text-books were found to deal only with the case in which the number under the root-sign is an integer, and to be for the most part fragmentary, and sometimes confusing. On this account it was thought right to publish for the use of students a logically arranged exposition of the subject in its general form, including the more elementary of the new results which had been obtained.

* The results of the investigation referred to have been communicated to the Royal Society of Edinburgh in a paper entitled "Continuants; a new special class of Determinants."

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1. Every arithmetical expression which is not exactly equal to an integer may be changed into the form of a continued fraction by performing the operations indicated, and finding by the ordinary rule the continued fraction equivalent to the decimal thus resulting.

In particular, any quadratic surd may be treated in this way,

$$e.g.\quad \sqrt{23} = 4.79583\ldots = 4 + \tfrac{79583}{100000} + \ldots$$

$$= 4 + \frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\ldots$$

2. In the case of a quadratic surd, however, it is much easier and much more instructive to proceed *directly* as follows:—

$$\sqrt{\frac{39}{5}} = \frac{\sqrt{195}}{5}$$

5) $\sqrt{195}$(2
10

$\sqrt{195} - 10$)5(

or, multiplying divisor and dividend by $\dfrac{\sqrt{195}+10}{5}$,

19) $\sqrt{195}+10$(1
19

$\sqrt{195} - 9$)19(

or, multiplying divisor and dividend by $\frac{\sqrt{195}+9}{19}$,

$$
\begin{array}{lllll}
 & 6)\ \sqrt{195}+9(3 & & & \\
 & \quad\ \ 18 & & & \\
 & \sqrt{195}-9)6(& & & \\
\text{or} & & 19)\sqrt{195}+9(1 & & \\
 & & \qquad\quad 19 & & \\
 & & \sqrt{195}-10)19(& & \\
 & \text{or} & & 5)\ \sqrt{195}+10(4 & \\
 & & & \qquad\quad 20 & \\
 & & & \sqrt{195}-10)5(& \\
 & & & \ldots\ldots\ldots\ldots &
\end{array}
$$

We do not proceed further, because the last line being exactly the same as the third line of the process, the operations henceforward will continually recur in the cycle already traversed.

$$\therefore \sqrt{\tfrac{39}{5}} = 2 + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \frac{1}{4} + \&c.$$

3. The continued fractions thus resulting have necessarily their numerators each equal to unity, and their partial denominators positive integers; and it is only continued fractions of this class which we mean to consider here. It must however be borne in mind that the same quadratic surd can be expressed as a continued fraction with other than unit-numerators, and this too in a variety of ways. For example it is easily shown that $\sqrt{23}$

$$\text{besides being } = 4 + \frac{1}{1} + \frac{1}{3} + \frac{1}{1} + \&c.$$

$$\text{also } = 4 + \frac{7}{8} + \frac{7}{8} + \frac{7}{8} + \&c.$$

$$= 3 + \frac{14}{6} + \frac{14}{6} + \frac{14}{6} + \&c.$$

$$= \ldots\ldots\ldots\ldots\ldots\ldots\ \ldots\ldots$$

4. *No quadratic surd can be expressed as a* terminating *continued fraction.*

For, otherwise, the last convergent being found, we should have a rational fraction equal to the surd, which is impossible.

5. If any positive number which is not an exact square be changed into the form $\frac{H}{G}$ where H and G are integers, the direct process of reduction of $\sqrt{\frac{H}{G}}$ to a continued fraction with unit numerators may be exhibited as follows:—

1. Suppose $H > G$ so that $\sqrt{\frac{H}{G}}$ or $\frac{\sqrt{HG}}{G} > 1$

$$G)\ \sqrt{HG}\ (A \quad \textit{i.e., highest integer} \text{ in } \frac{\sqrt{HG}}{G}$$
$$\underline{AG\qquad}$$
$$\sqrt{HG} - AG)\ G\ ($$

or,

Call this divisor, D_1,
$$H - A^2 G)\ \sqrt{HG} + AG\ (Q_1$$
$$\underline{Q_1 D_1} \quad \text{Write } R_2 \text{ for } Q_1 D_1 - AG$$
$$\sqrt{HG} - R_2)\ D_1\ ($$

or,

Call this divisor, d_2,
$$\frac{HG - R_2^2}{D_1}\ )\ \sqrt{HG} + R_2\ (Q_2$$
$$\underline{Q_2 d_2} \quad \text{Write } r_3 \text{ for}$$
$$\sqrt{HG} - r_3) d_2 (\quad Q_2 d_2 - R_2$$

or,

Call this divisor, d_3,
$$\frac{HG - r_3^2}{d_2}\ )\ \sqrt{HG} + r_3\ (Q_3$$
$$\underline{Q_3 d_3}$$
$$\dots\dots\dots\dots\dots\dots$$

the result being

$$\sqrt{\frac{H}{G}} = A + \frac{1}{Q_1 +}\ \frac{1}{Q_2 +}\ \frac{1}{Q_3 +} \dots$$

where $Q_1, Q_2, Q_3, \dots\dots$ are from the nature of the process all positive integers. For clearness in what follows, other capital letters have been used wherever it is perfectly evident that the numbers to be represented are integral.

2. Suppose $H < G$, then $\sqrt{\frac{H}{G}} = \frac{1}{\sqrt{\frac{G}{H}}}$, and the continued fraction $= \sqrt{\frac{G}{H}}$ is found in the way shown above, and substituted for it.

6. The process involves an infinite series of divisors, dividends, and quotients. The quotients are obviously *partial quotients,* each with a complete quotient corresponding. Every dividend is the sum of a surd and a rational term, which rational terms may be spoken of shortly as the *rational dividends.*

Clearly, if at any stage we should have simultaneously the same divisor and rational dividend as have before simultaneously occurred, the process may be discontinued, as thenceforward it would be but an unending repetition of what had already been gone through.

7. The divisors G, D_1, d_2, d_3......, the rational dividends 0, AG, R_2, r_3......and the partial quotients A, Q_1, Q_2, Q_3...... are evidently related to each other in the following manner:—

$$Q_n = \text{highest integer in } \frac{\sqrt{HG} + r_n}{d_n} \quad \ldots\ldots\ldots(1)$$

$$r_{n+1} = Q_n d_n - r_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

$$d_{n+1} = \frac{HG - r_{n+1}^2}{d_n} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $n = 0, 1, 2, 3\ldots\ldots$

8. *All the divisors and rational dividends are integral.*

From (2) and (3):—

$$d_{n+1} = \frac{HG - (Q_n d_n - r_n)^2}{d_n}$$

$$= \frac{HG - r_n^2}{d_n} - Q_n^2 d_n + 2\ Q_n r_n$$

$$\therefore (3) \qquad = d_{n-1} - Q_n^2 d_n + 2\ Q_n r_n$$

This equation and (2) show that if d_{n-1}, d_n, r_n be integers, then r_{n+1}, d_{n+1} must also be integers. But the first two divisors and second rational dividend are known to be integral, therefore the third divisor and third rational dividend must be integral, and so on.

N.B.—Instead of D_1, R_2, Q_1, Q_2 ...... we shall now for convenience use the corresponding small letters, it being understood throughout what follows that, unless otherwise stated, all numbers represented by letters are integral.

9. *The divisors are all positive.*

Suppose d_n positive, then—

(1) $\quad q_n\, d_n < \sqrt{HG} + r_n$

$\therefore$ (2) $\quad \sqrt{HG} > r_{n+1}$

$\therefore$ (3) $\quad d_n d_{n+1} > 0$

So that if any one of the divisors d_n be positive, so also must the next d_{n+1}. But the first and second are known to be positive, therefore the third is positive, and so on.

10. *No rational dividend can be greater than the highest integer in the surd-dividend.*

(1) $\quad q_n\, d_n < \sqrt{HG} + r_n$

$\therefore$ (3) $\quad r_{n+1} + r_n < \sqrt{HG} + r_n$

$\therefore$ $\quad r_{n+1} < \sqrt{HG}$...............(4)

$\therefore$ $\quad \overline{\overline{<}}$ highest integer in $\sqrt{HG}$, which integer let us in future indicate by I.

11. *The rational dividends are all positive.*

Suppose r_n positive, then—

(4) $\quad \sqrt{HG} > r_n$

$\therefore$ (1) $\quad q_n\, d_n > r_n$

$\therefore$ (2) $\quad r_{n+1} > 0$

So that if any one of the rational dividends be positive, so also must the next. But the first and second are known to be positive, therefore the third is positive, and so on.

12. *No divisor or partial quotient can be greater than double the highest integer in the surd-dividend.*

For if either q_n or d_n were $> 2\,I$ then (2) would $r_n + r_{n+1}$ be $> 2\,I$ contrary to § 10.

13. *A divisor double of the highest integer in the surd-dividend can only occur when the number under the root-sign is less by unity than an exact square.*

For, putting $d_n = 2I$ in (2) we have $2I\,q_n = r_n + r_{n+1}$

But (§ 10) $r_n + r_{n+1}$ is not $> 2I$

$$\therefore \qquad q_n = 1$$

$$\text{and} \qquad r_n = I = r_{n+1}$$

$$\therefore \quad (3) \qquad \frac{HG - I^2}{2I} = d_{n+1} \text{ and } \therefore \overline{\overline{>}}\ 1$$

But I being the highest integer in $\sqrt{HG}$,

$$HG \overline{\overline{<}}\ I^2 + 2I$$

$$\therefore \quad \frac{HG - I^2}{2I} \overline{\overline{<}}\ 1$$

Whence necessarily, $\dfrac{HG - I^2}{2I} = 1$

$$\text{and} \quad \therefore \quad HG = (I+1)^2 - 1$$

14. *If any divisor be subtracted from the next but one, and the rational dividend corresponding to the latter be subtracted from the one preceding it, the first of these remainders is exactly divisible by the second, the quotient being the partial quotient corresponding to the last-mentioned rational dividend.*

$$(3) \qquad d_n\, d_{n+1} = HG - r^2_{n+1}$$

$$\text{and} \qquad d_{n-1}\, d_n = HG - r_n{}^2$$

$$\therefore \quad (d_{n+1} - d_{n-1})\, d_n = r^2_n - r^2_{n+1}$$

$$= (r_n + r_{n+1})\,(r_n - r_{n+1})$$

$$\text{But (2)} \qquad q_n d_n = r_n + r_{n+1}$$

$$\therefore \quad \frac{d_{n+1} - d_{n-1}}{q_n} = r_n - r_{n+1}$$

$$\text{and} \quad \frac{d_{n+1} - d_{n-1}}{r_n - r_{n+1}} = q_n \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

15. *The sum of any divisor and the following rational dividend is greater than the surd-dividend, and the sum of any divisor except the first and the corresponding rational dividend is greater than the surd-dividend.*

From the nature of the process,

$$\sqrt{HG} - r_{n+1} < d_n$$

$$\therefore \quad d_n + r_{n+1} > \sqrt{HG} \quad \ldots\ldots\ldots\ldots\ldots (6)$$

$$(2) \quad r_{n-1} + r_n = q_{n-1}\, d_{n-1}$$

$$\therefore \; (4) \quad \sqrt{HG} + r_n > q_{n-1}\, d_{n-1}$$

$$> d_{n-1} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7)$$

Now, except when $n = 0$ we have from (3)

$$d_n = \frac{HG - r_{n+1}^2}{d_{n-1}}$$

$$= \frac{\sqrt{HG} + r_n}{d_{n-1}} \left(\sqrt{HG} - r_n \right)$$

$$\therefore \; (7) \quad > \sqrt{HG} - r_n$$

$$\therefore \quad \text{when } n > 0,\; d_n + r_n > \sqrt{HG} \quad \ldots\ldots\ldots\ldots (8)$$

16. *The difference between any two rational dividends except the first is less than either the divisor corresponding to the subtrahend or the divisor preceding that.*

Of any two rational dividends r_m, r_n let r_m be the greater.

$$(8) \quad \text{when } n > 0,\; d_n > \sqrt{HG} - r_n$$

$$\text{and } (6) \quad d_{n-1} > \sqrt{HG} - r_n$$

$$\text{and } (4) \quad r_m < \sqrt{HG}$$

$$\therefore \quad \text{when } n > 0,\; r_m - r_n < d_n \text{ and } < d_{n-1} \ldots\ldots (9)$$

17. Def^{n}. A continued fraction having a set of constituents which recurs uninterruptedly, ad infinitum, is called a *recurring* continued fraction.

In writing such fractions we adopt the notation used in connection with recurring decimals in Arithmetic, only using an asterisk instead of a dot.

Thus $3 + \frac{1}{1+}\, \frac{1}{\underset{*}{4}+}\, \frac{1}{5+}\, \frac{1}{\underset{*}{2}}$ is meant to denote

$$3 + \frac{1}{1+}\, \frac{1}{4+}\, \frac{1}{5+}\, \frac{1}{2+}\, \frac{1}{4+}\, \frac{1}{5+}\, \frac{1}{2+}\, \frac{1}{4+} \;\&\text{c.}$$

The operation of the rule for calculating the numerators and denominators of the convergents of a continued fraction which has unit-numerators, we denote by K. Thus $K(a) = a$, $K, (a\, b)$

$= ab + 1$, $K(a, b, c) = abc + c + a$, &c., and the convergents of the above continued fraction are $\frac{K(3,1)}{K(1)}$, $\frac{K(3,1,4)}{K(1,4)}$, $\frac{K(3,1,4,5)}{K(1,4,5)}$ &c.

18. *The continued fraction with unit-numerators and positive integral partial-denominators, found as an equivalent of any quadratic surd, is always a recurring continued fraction, the end of the cycle being reached before more than* $2\,I^2$ *partial quotients have been found.*

For no divisor can be greater than $2\,I$ (§ 12) and no rational dividend can be greater than I (§ 10); so that by the time that $2\,I \times I$ divisions have been performed every possible divisor must have occurred along with every possible rational dividend; and consequently, if a divisor and rational dividend have not already for the second time appeared simultaneously—thus necessitating a cycle in the operations of dividing—such must take place before more than $2\,I^2$ divisions have been performed.

19. *In every case the cycle begins with the second operation of division.*

Suppose that the $(s+1)^{\text{th}}$ operation of division is exactly the same as a previous one, the $(e+1)^{\text{th}}$ say, so that

$$d_s = d_e \; r_s = r_e \text{ and } \therefore \; q_s = q_e$$

From (3) $d_{s-1}\, d_s = HG - r_s^2$ and $d_{e-1}\, d_e = HG - r_e^2$

$$\therefore \text{ (hyp.)} \qquad d_{s-1} = d_{e-1} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Again (2) $r_{s-1} + r_s = q_{s-1}\, d_{s-1}$ and $r_{e-1} + r_e = q_{e-1}\, d_{e-1}$

$\therefore$ by subtraction we have (10) and (hyp.)

$$(r_{s-1} \sim r_{e-1}) \div d_{e-1} = q_{s-1} \sim q_{e-1}$$

But (§ 16) when $e > 1$

$$r_{s-1} \sim r_{e-1} < d_{e-1}$$

The only possible result therefore is

e being > 1, $r_{s-1} = r_{e-1}$ and $q_{s-1} = q_{e-1}$

e being $= 1$, $r_{e-1} = 0$, consequently r_{s-1} is not $= r_{e-1}$ and q_{s-1} is not $= q_{e-1}$

We have thus shown that some one of the operations of division beyond the second being known to recur, its predecessor must also recur; and going backwards in this manner we see that in

every case the *second* operation of division must recur. Further, the second operation being known to recur we have proved that the first cannot, therefore the cycle commences with the second operation.

Thus, up to this point we have learned that every quadratic surd when expressed as a continued fraction with unit-numerators takes the form—

$$A + \frac{1}{\overset{*}{q_1} +}\,\frac{1}{q_2 +}\,\frac{1}{q_3 +}\cdots + \frac{1}{\overset{*}{q_\iota} +}\cdots$$

$$\text{or}\quad \frac{1}{A +}\,\frac{1}{\overset{*}{q_1} +}\,\frac{1}{q_2 +}\,\frac{1}{q_3 +}\cdots + \frac{1}{\overset{*}{q_\iota} +}\cdots$$

according as the surd is greater or less than unity.

20. Defn. The first partial quotient of all may thus be spoken of as the *unique partial quotient.*

21. *The last member of the cycle of partial quotients is double of the unique partial quotient; the last member of the cycle of divisors is equal to the first divisor of all; and the last member of the cycle of rational dividends is equal to the first member of the cycle.*

From (3) $\qquad d_{\iota+1}\, d_\iota = HG - r_{\iota+1}^2$

But (§ 19) $\quad d_{\iota+1} = d_1$ and $r_{\iota+1} = r_1$ and $\therefore = AG$

$\therefore \qquad d_1\, d_\iota = HG - A^2G^2 = d_1\, G \quad$ (§ 5)

$\therefore \qquad d_\iota = G \;\; i.e.\;\; d_0$

Again, $\qquad q_{\iota-1}\, d_{\iota-1} = r_\iota + r_{\iota-1} = 2\, r_\iota + r_{\iota-1} - r_\iota$

$\therefore$ (§ 16) $\qquad < 2r_\iota + d_\iota$

$\therefore \qquad d_{\iota-1} < 2r_\iota + G$

and $\qquad \dfrac{d_{\iota-1}}{G} < 2\,\dfrac{r_\iota}{G} + 1$ ....(11)

Now (3) $\quad Gd_{\iota-1} = HG - r_\iota^2$, r_ι being thus a multiple of G,

$$\therefore \qquad \left(\frac{r_\iota}{G}\right)^2 + \frac{d_{\iota-1}}{G} = \frac{H}{G}.$$

But from (11) we see that the highest integer in the square

root of the first member is $\frac{r_l}{G}$ and the highest integer in $\sqrt{\frac{H}{G}}$ is of course A.

$$\therefore \qquad r_l = A\,G \quad i.e. \quad r_1.$$

Lastly, since (2) $\qquad q_l\, d_l = r_l + r_{l+1}$

$$\therefore \qquad q_l\, G = A\, G + A\, G$$

$$\therefore \qquad q_l = 2\, A.$$

22. *If the last member of the cycle of partial quotients be omitted, what remains is the same when read backwards as when read forwards; the same is true of the cycle of divisors; and of the cycle of rational dividends when the last member is included.* That is, if l be the number of members in the cycle, then

$$r_l = r_1 \text{ (already proved)}$$

$$d_{l-1} = d_1 \quad q_{l-1} = q_1 \quad r_{l-1} = r_2$$

$$d_{l-2} = d_2 \quad q_{l-2} = q_2 \quad r_{l-2} = r_3$$

$$\dots\dots\dots \quad \dots\dots\dots \quad \dots\dots\dots$$

For (3) $\qquad d_l\, d_{l-1} = HG - r_l^2$

$\therefore$ (§ 21) $\quad G\, d_{l-1} = HG - A^2G^2$

$$\therefore \qquad d_{l-1} = H - A^2G = d_1 \qquad (\S\ 5)$$

From (2) $\qquad q_{l-1} = (r_{l-1} + r_l) \div d_{l-1}$

$\therefore$ (§ 21) $\qquad = (r_{l-1} + r_1) \div d_{l-1}$

But $\qquad q_1 = (r_1 + r_2) \div d_1$

$$\therefore \qquad q_{l-1} \sim q_1 = (r_{l-1} \sim r_2) \div d_{l-1} \text{ (or } d_1)$$

and (§ 16) $\quad r_{l-1} \sim r_2 < d_{l-1}$ or d_1

$$\therefore \qquad q_{l-1} = q_1 \text{ and } r_{l-1} = r_2$$

If we now suppose that

$$d_{l-h} = d_h,\ q_{l-h} = q_h \text{ and } r_{l-h} = r_{h+1} \dots\dots\dots (12)$$

it is evident that we may, by the above method, prove

$$d_{l-(h+1)} = d_{h+1},\ q_{l-(h+1)} = q_{h+1} \text{ and } r_{l-(h+1)} = r_{h+2}$$

But (12) is true when $h = 1$ $\therefore$ it must also be true when $h = 2$, and so on.

Thus, every quadratic surd when expressed as a continued

fraction with unit-numerators takes the form

$$A+\overset{*}{\frac{1}{q_1+}}\,\frac{1}{q_2+}\cdots\cdots+\frac{1}{q_2+}\,\frac{1}{q_1+}\,\overset{*}{\frac{1}{2A+}}\cdots$$

or its reciprocal.

23. All the terms of the cycle of partial quotients except the last constitute what may be called the *symmetric portion* of the cycle, and by the *middle term* or *terms* of the cycle is meant the middle term or terms of the symmetric portion of the cycle: similarly in the case of the cycle of divisors.

24. *In the cycle of rational dividends no two consecutive terms can be equal without being the middle terms of the cycle; and the divisor and partial quotient corresponding to the former of the two must each be the middle term of its own cycle.*

For suppose $\qquad r_{n+1}=r_n \ldots\ldots\ldots\ldots\ldots\ldots(13)$

$$\text{then (3)}\quad \left.\begin{aligned} d_{n+1}\,d_n &= HG-r_{n+1}^2 \\ \text{and}\quad d_n\,d_{n-1} &= HG-r_n^2 \end{aligned}\right\}$$

$$\therefore\ (13)\qquad d_{n+1}=d_{n-1}\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

$$\text{From (2)}\quad \left.\begin{aligned} q_{n+1}\,d_{n+1} &= r_{n+1}+r_{n+2} \\ \text{and}\quad q_{n-1}\,d_{n-1} &= r_{n-1}+r_n \end{aligned}\right\}$$

$$\therefore\ (13)\text{ and }(14)\quad q_{n+1}\sim q_{n-1}=(r_{n+2}\sim r_{n-1})\div d_{n-1}\ (\text{or } d_{n+1})$$

$$(\S\ 16)\qquad\qquad <1$$

$$\therefore\quad q_{n+1}=q_{n-1}\text{ and } r_{n+2}=r_{n-1}\ldots\ldots\ldots(15)$$

Again from (14) and (15) we may show that

$$r_{n+3}=r_{n-2}$$

and so on. Therefore r_{n+1} and r_n are the middle terms of their cycle. As in the two other cycles one of the terms (§ 23) is not reckoned in finding the middle term, it is clear that the middle term of each is that corresponding to r_n.

25. The preceding theorem is a result of practical importance.

Ex. Expand $\sqrt{\dfrac{17}{12}}$ as a continued fraction with unit-numerators.

$12)\ \sqrt{204}(1$
12
$\overline{\sqrt{204}-12)12(}$
$5)\ \sqrt{204}+12(5$
25
$\overline{\sqrt{204}-13)5(}$
$7)\ \sqrt{204}+13(3$
21
$\overline{\sqrt{204}-8)7(}$
$20)\ \sqrt{204}+8(1$
20
$\overline{\sqrt{204}-12)20(}$
$3)\ \sqrt{204}+12(8$
24
$\overline{\sqrt{204}-12}$
....

Observing here that $r_5 = 12 = r_4$ we proceed no further, since we know that q_4 *i.e.* 8 must be the middle term of the cycle of partial quotients. We have thus

$$\sqrt{\frac{17}{12}} = 1 + \underset{*}{\frac{1}{5}} + \frac{1}{3} + \frac{1}{1} + \frac{1}{8} + \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \underset{*}{\frac{1}{2}} + \ldots.$$

26. *If two consecutive divisors be alike and also the two corresponding partial quotients, they must be the middle terms of their respective cycles.*

For suppose $\qquad d_n = d_{n+1}$ and $q_n = q_{n+1}$

$$\text{then } q_n\, d_n = q_{n+1}\, d_{n+1}$$

$$\therefore\ (2) \qquad r_n + r_{n+1} = r_{n+1} + r_{n+}$$

$$\therefore\ \ r_n = r_{n+2} \ldots\ldots\ldots\ldots (16)$$

$$\text{Hence (3)} \quad d_n\, d_{n-1} = d_{n+2}\, d_{n+1}$$

$$\therefore\ (\text{hyp.}) \quad d_{n-1} = d_{n+2} \ldots\ldots\ldots\ldots (17)$$

$$\left.\begin{aligned} &\text{But } (2)\ \ q_{n-1}\, d_{n-1} = r_{n-1} + r_n \\ &\text{and } \quad q_{n+2}\, d_{n+2} = r_{n+2} + r_{n+3} \end{aligned}\right\}$$

$\therefore$ (16) and (17) $\quad q_{n-1} \sim q_{n+2} = (r_{n-1} \sim r_{n+3}) \div d_{n+2}$ (or d_{n-1})

(§ 16) $\qquad < 1$

$$\therefore \quad q_{n-1} = q_{n+2} \text{ and } r_{n-1} = r_{n+3}$$

Again, from these results we may show that

$$d_{n-2} = d_{n+3} \text{ and } q_{n-2} = q_{n+3}$$

and so on. Therefore d_n and d_{n+1}, q_n and q_{n+1} are the middle terms of their respective cycles.

27. The following is an illustrative example of the practical value of the preceding theorem.

Ex. Expand $\sqrt{\dfrac{1}{61}}$ as a continued fraction with unit-numerators.

1) $\sqrt{61}$(7
$\quad 7$
$\sqrt{61}-7$)1(
$\quad$ 12) $\sqrt{61}+7$(1
$\qquad 12$
$\quad \sqrt{61}-5$)12(
$\qquad$ 3) $\sqrt{61}+5$(4
$\qquad\quad 12$
$\qquad \sqrt{61}-7$)3(
$\qquad\quad$ 4) $\sqrt{61}+7$(3
$\qquad\qquad 12$
$\qquad\quad \sqrt{61}-5$)4(
$\qquad\qquad$ 9) $\sqrt{61}+5$(1
$\qquad\qquad\quad 9$
$\qquad\qquad \sqrt{61}-4$)9(
$\qquad\qquad\quad$ 5) $\sqrt{61}+4$(2
$\qquad\qquad\qquad 10$
$\qquad\qquad\quad \sqrt{61}-6$)5(
$\qquad\qquad\qquad$ 5) $\sqrt{61}+6$(2
$\qquad\qquad\qquad$ ....

Here we have two consecutive partial quotients equal and also the two corresponding divisors equal, and we proceed no further,

knowing that these must be the middle terms of their respective cycles. The result $\therefore$ is

$$\sqrt{\frac{1}{61}} = \frac{1}{7+}\ \underset{*}{\frac{1}{1+}}\ \frac{1}{4+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{4+}\ \frac{1}{1+}\ \underset{*}{\frac{1}{14+}} \ldots$$

28. Since $\dfrac{K(A, q_1 \ldots q_n)}{K(q_1 \ldots q_n)}$ is more or less approximately $= \sqrt{\dfrac{H}{G}}$, it is clear that $G\,K(A, q_1 \ldots q_n)^2 - H\,K(q_1 \ldots q_n)^2$ must be a comparatively small quantity; and, similarly, since $\dfrac{K(A, q_1 \ldots q_{n-1})}{K(q_1 \ldots q_{n-1})}$ and $\dfrac{K(A, q_1 \ldots q_n)}{K(q_1 \ldots q_n)}$ are approximations to $\sqrt{\dfrac{H}{G}}$ and their product therefore approximately equal to $\dfrac{H}{G}$, it follows that $GK(A, q_1 \ldots q_{n-1})$ $K(A, q_1 \ldots q_n) - H\,K(q_1 \ldots q_{n-1})\ K(q_1 \ldots q_n)$ must be comparatively small. These considerations serve as a preparation for, and an introduction to, two theorems which elegantly express any divisor and rational dividend in terms of the preceding partial quotients and the given number.

29. $(-1)^{n+1}\, d_{n+1} = G\,K(A, q_1 \ldots q_n)^2 - H\,K(q_1 \ldots q_n)^2$

$(-1)^n r_{n+1} = GK(A, q_1 \ldots q_n)\,K(A, q_1 \ldots q_{n-1}) - HK(q_1 \ldots q_n)\,K(q_1 \ldots q_{n-1})$

If at any stage in the calculation of the convergent we use the *complete* quotient instead of the corresponding partial quotient the result is not a *convergent* to the given surd, but an expression exactly *equal* to it. Using, then, the $(n+1)^{\text{th}}$ complete quotient, we have

$$\frac{\dfrac{\sqrt{HG} + r_{n+1}}{d_{n+1}}\ K(A, q_1 \ldots q_n) + K(A, q_1 \ldots q_{n-1})}{\dfrac{\sqrt{HG} + r_{n+1}}{d_{n+1}}\ K(q_1 \ldots q_n) + K(q_1 \ldots q_{n-1})} = \frac{\sqrt{HG}}{G}.$$

Clearing of fractions and equating the rational terms of the one side to those of the other, and the irrational to the irrational, there results

$$\left.\begin{aligned} r_{n+1}\,K(A, q_1 \ldots q_n) + d_{n+1}\,K(A, q_1 \ldots q_{n-1}) &= HK(q_1 \ldots q_n) \ldots\ldots (18) \\ r_{n+1}\,K(q_1 \ldots q_n) + d_{n+1}\,K(q_1 \ldots q_{n-1}) &= GK(A, q_1 \ldots q_n) \ldots (19) \end{aligned}\right\}$$

Eliminating from these equations r_{n+1} and d_{n+1} in succession, we have

$$d_{n+1} = \frac{HK(q_1 \ldots q_n)^2 - GK(A, q_1 \ldots q_n)^2}{K(A, q_1 \ldots q_{n-1})\,K(q_1 \ldots q_n) - K(A, q_1 \ldots q_n)\,K(q_1 \ldots q_{n-1})}$$

$$\therefore (-1)^{n+1}\, d_{n+1} = GK(A, q_1 \ldots q_n)^2 - HK(q_1 \ldots q_n)^2 \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

and $(-1)^n\, r_{n+1} = GK(A, q_1 \ldots q_{n-1}) - HK(q_1 \ldots q_n)\, K(q_1 \ldots q_{n-1}) \ldots (21)$

The most important particular case of these equations is that in which d_{n+1} is the last divisor of the cycle. (18), (19) and (20) then become

$$A\,GK(A, q_1 \ldots q_1) + GK(A, q_1 \ldots q_2) = HK(q_1 \ldots q_1) \ldots\ldots\ldots (22)$$

$$A\,K(q_1 \ldots q_1) \quad + \quad K(q_1 \ldots q_2) \quad = K(A, q_1 \ldots q_1) \ldots\ldots\ldots (23)$$

$$GK(A, q_1 \ldots q_1)^2 + HK(q_1 \ldots q_1)^2 \quad = \pm\, G \ldots\ldots\ldots\ldots\ldots\ldots (24)$$

the last of which is notable as affording a solution in positive integers of the indeterminate equation $x^2 - N\,y^2 = 1$ where N is any given positive integer or fraction.

30. $\dfrac{K(A, q_1 \ldots q_m)^2}{K(q_1 \ldots q_m)^2}$ and $\dfrac{K(A, q_1 \ldots q_n)^2}{K(q_1 \ldots q_n)}$ being each approximately equal to $\dfrac{H}{G}$, so also must $\dfrac{K(A, q_1 \ldots q_m)^2 + K(A, q_1 \ldots q_n)^2}{K(q_1 \ldots q_m)^2 + K(q_1 \ldots q_n)^2}$ which in magnitude lies between them. For particular values of m and n this approximation is perfect as we shall now show.

31. *In the case of a cycle of partial quotients which consists of* $2z + 1$ *terms* $\dfrac{K(A \ldots q_{z-s-1})^2 + K(A \ldots q_{z+s})^2}{K(q_1 \ldots q_{z-s-1})^2 + K(q_1 \ldots q_{z+s})^2} = \dfrac{H}{G}$, *where* s *is zero or any integer less than* z.

For, there being $2z + 1$ terms in the cycles, $d_z = d_{z+1}$, $d_{z-1} = d_{z+2}$ and generally $d_{z-s} = d_{z+s+1}$

$$\text{and } \therefore\ (-1)^{z-s}\, d_{z-s} = -(-1)^{z+s+1}\, d_{z+s+1}$$

therefore (§ 29.)

$$GK(A \ldots q_{z-s-1})^2 - HK(q_1 \ldots q_{z-s-1})^2 = HK(q_1 \ldots q_{z+s})^2 - GK(A \ldots q_{z+s})^2$$

whence the theorem.

There are two particular cases worthy of notice, viz. by taking $s=0$ and $=z-1$.

$$\frac{K(A\ldots q_{z-1})^2+K(A\ldots q_z)^2}{K(q_1\ldots q_{z-1})^2+K(q_1\ldots q_z)^2}=\frac{H}{G}\ldots\ldots\ldots\ldots\ldots(25)$$

$$\frac{A^2+K(A\ldots q_{2z-1})^2}{1+K(q_1\ldots q_{2z-1})^2}=\frac{H}{G}\ldots\ldots\ldots\ldots\ldots(26)$$

32. *In the case of a cycle of partial quotients which consists of* $2z$ *terms*

$$\frac{K(A\ldots q_{z-s-2})\,K(A\ldots q_{z-s-1})+K(A\ldots q_{z+s-1})\,K(A\ldots q_{z+s})}{K(q_1\ldots q_{z-s-2})\,K(q_1\ldots q_{z-s-1})+K(q_1\ldots q_{z+s-1})\,K(q_1\ldots q_{z+s})}=\frac{H}{G}$$

where s *is zero or any integer less than* z − 1.

For there being $2z$ terms in the cycles it follows that $r_z=r_{z-1}$, $r_{z-1}=r_{z+2}$ and generally $r_{z-s}=r_{z+s+1}$

and $\therefore\ (-1)^{z-s-1}\,r_{z-s}=-(-1)^{z+s}\,r_{z+s+1}$

$$\therefore (\S 29)\ GK(A\ldots q_{z-s-2})\,K(A\ldots q_{z-s-1})-HK(q_1\ldots q_{z-s-2})K(q_1\ldots q_{z-s-1})$$
$$=-GK(A\ldots q_{z+s-1})\,K(A\ldots q_{z+s})\ +HK(q_1\ldots q_{z+s-1})K(q_1\ldots q_{z+s})$$

whence the theorem.

An important particular case is that in which $s=0$, viz.:—

$$\frac{K(A\ldots q_{z-1})\,\{K(A\ldots q_{z-2})+K(A\ldots q_z)\}}{K(q_1\ldots q_{z-1})\,\{K(q_1\ldots q_{z-2})+K(q_1\ldots q_z)\}}=\frac{H}{G}\ldots\ldots\ldots(27)$$

33. *If instead of using the last partial quotient in calculating in the ordinary way the convergent corresponding to it we use its half, the numerator of the result divided by the preceding denominator is exactly equal to the given number whose square root was to be found.*

First, when the number of terms in the cycle is odd, $2z+1$, we know (25) that

$$\frac{H}{G}=\frac{K(A\ldots q_{z-1})^2+K(A\ldots q_z)^2}{K(q_1\ldots q_{z-1})^2+K(q_1\ldots q_z)^2}$$

$$\therefore\quad \text{(App.)}\qquad =\frac{K(A,q_1\ldots q_z,\ q_z\ldots q_1,\ A)}{K(q_1\ldots q_z,\ q_z\ldots q_1)}\ \text{as was to be proved.}$$

Second, when the number of terms in the cycle is even, $2z$, we know (27) that

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$$\frac{H}{G}=\frac{K(A\ldots q_z)\,K(A\ldots q_{z-1})+K(A\ldots q_{z-1})\,K(A\ldots q_{z-2})}{K(q_1\ldots q_z)\,K(q_1\ldots q_{z-1})+K(q_1\ldots q_{z-1})\,K(q_1\ldots q_{z-2})},\ \therefore\ \text{(App.)}$$

$$=\frac{K(A,\,q_1\ldots q_{z-1},\,q_z,\,q_{z-1}\ldots q_1,\,A)}{K(q_1\ldots q_{z-1},\,q_z,\,q_{z-1}\ldots q_1)},\quad \text{as was to be proved.}$$

34. *In the case of a cycle of partial quotients which consists of* 2z *terms*

$$\frac{K(A\ldots q_z)^2-K(A\ldots q_{z-2})^2}{K(q_1\ldots q_z)^2-K(q_1\ldots q_{z-2})^2}=\frac{H}{G}.$$

For $q_z\,K(A\ldots q_{z-1},\,q_z,\,q_{z-1}\ldots A)=K(A\ldots q_z)^2-K(A\ldots q_{z-2})^2$

and $q_z\,K(q_1\ldots q_{z-1},\,q_z,\,q_{z-1}\ldots q_1)=K(q_1\ldots q_z)^2-K(q_1\ldots q_{z-2})^2$

and thus the theorem follows from the preceding.

35. The fact (§ 18) that the partial quotients recur in a cycle might lead us to expect a simplification of some kind in the calculation of the convergents, and the further fact (§ 22) of the peculiar nature of the cycle induces the same expectation. The way in which this is realized will be explained in what immediately follows.

36. *If the cycle of partial quotients consist of* 2n *terms and the first* n + 1 *convergents have been calculated, the* $2n^{th}$ *convergent may be found without calculating the intermediate ones.*

Let $$\sqrt{\frac{H}{G}}=A+\frac{1}{\underset{*}{q_1}+}\,\frac{1}{q_2+}\ldots+\frac{1}{q_n+}\ldots+\frac{1}{q_2+}\,\frac{1}{q_1+}\,\frac{1}{\underset{*}{2A}+}\ldots$$

then the $2n^{th}$ convergent is

$$\frac{K(A,\,q_1,\ldots\ldots q_n\ldots\ldots q_1)}{K(q_1,\ldots\ldots q_n\ldots\ldots q_1)}$$

But this $$=\frac{K(A,\,q_1\ldots q_n)\,K(q_1\ldots q_{n-1})+K(A,q_1\ldots q_{n-1})K(q_1\ldots q_{n-2})}{K(q_1\ldots q_{n-1})\,\{K(q_1\ldots q_n)+K(q_1\ldots q_{n-2})\}}$$

Now if the first $n+1$ convergents have been calculated, $K(A,\,q_1\ldots q_n)$, $K(A,\,q_1\ldots q_{n-1})$, $K(q_1\ldots q_n)$, $K(q_1\ldots q_{n-1})$, $K(q_1\ldots q_{n-2})$ are all known, and thus the theorem is proved.

37. *If the cycle of partial quotients consist of* 2n + 1 *terms and the first* n + 1 *convergents have been calculated, the* $(2n+1)^{th}$ *convergent may be found without calculating the intermediate ones.*

Let $\sqrt{\dfrac{H}{G}} = A + \dfrac{1}{\overset{*}{q_1}} + \dfrac{1}{q_2} + \ldots + \dfrac{1}{q_n} + \dfrac{1}{q_n} + \ldots + \dfrac{1}{q_2} + \dfrac{1}{q_1} + \dfrac{1}{\overset{*}{2A}} + \ldots$

then the $(2n+1)^{\text{th}}$ convergent is

$$\frac{K(A,\, q_1 \ldots q_n,\, q_n \ldots q_1)}{K(q_1 \ldots q_n,\, q_n \ldots q_1)}$$

$$\text{which } = \frac{K(A,\, q_1 \ldots q_n)\,K(q_1 \ldots q_n) + K(A,\, q_1 \ldots q_{n-1})\,K(q_1 \ldots q_{n-1})}{K(q_1 \ldots q_n)^2 + K(q_1 \ldots q_{n-1})^2}$$

and is thus dependent upon convergents preceding the $(n+2)^{th}$.

38. *If the cycle of partial quotients consist of* z *terms and the* mzth *convergent has been calculated, the* 2mzth *convergent may be found without calculating the intermediate ones.*

Let the cycle of z terms be $q_1, q_2, \ldots q_2, q_1, 2A$ then the $2z^{th}$ convergent is

$$\frac{K(A,\, q_1 \ldots q_1,\, 2A,\, q_1 \ldots q_1)}{K(q_1 \ldots q_1,\, 2A,\, q_1 \ldots q_1)} \text{ which}$$

$$= \frac{K(A,\, q_1 \ldots q_1,\, 2A)\,K(q_1 \ldots q_1) + K(A,\, q_1 \ldots q_1)\,K(q_1 \ldots q_2)}{K(q_1 \ldots q_1)\,\{K(q_1 \ldots q_1,\, 2A) + K(q_1 \ldots q_2)\}}$$

$$= \frac{\{2AK(A \ldots q_1) + K(A \ldots q_2)\}K(q_1 \ldots q_1) + K(A \ldots q_1)K(q_1 \ldots q_2)}{K(q_1 \ldots q_1)\{2AK(q_1 \ldots q_1) + 2K(q_1 \ldots q_2)\}}$$

$$= \frac{\{AK(A..q_1) + K(A..q_2)\}K(q_1..q_1) + K(A..q_1)\{AK(q_1..q_1) + K(q_1..q_2)\}}{2K(q_1 \ldots q_1)K(A,\, q_1 \ldots q_1)}$$

$$= \frac{HK(q_1 \ldots q_1)^2 + GK(A \ldots q_1)^2}{2\,GK(q_1 \ldots q_1)K(A,\, q_1 \ldots q_1)} \text{ by (22)}$$

The theorem is thus true when $m = 1$, and it is clear that the mode of proof holds for all values of m. (Observe that a convergent found in this way is not in its simplest form, G being a factor of both numerator and denominator.)

39. From these theorems we see that in no case is it necessary to continue the ordinary method of calculating the convergents after the middle of the cycle of partial quotients has been reached.

EXAMPLES.

Ex. 1. $\sqrt{\dfrac{18}{7}} = 1 + \dfrac{1}{\underset{*}{1}} + \dfrac{1}{1} + \dfrac{1}{1} + \dfrac{1}{1} + \dfrac{1}{10} + \dfrac{1}{1} + \dfrac{1}{1} + \dfrac{1}{1} + \dfrac{1}{1} + \dfrac{1}{\underset{*}{2}} + \ldots$

Here the first *six* convergents are

$$\frac{1}{1},\ \frac{2}{1},\ \frac{3}{2},\ \frac{5}{3},\ \frac{8}{5},\ \frac{85}{53}$$

$$\therefore\ (\S\ 36)\ \text{the } \textit{tenth} \text{ convergent} = \frac{85 \times 5 + 8 \times 3}{5(3+53)}$$

$$= \frac{449}{280}$$

$$\therefore\ (\S\ 38)\ \text{the } \textit{twentieth} = \frac{18(280)^2 + 7(449)^2}{2.7.280.449}$$

$$= \frac{403201}{251440}$$

$$\ldots\ldots\ldots \text{ the } \textit{fortieth} = \frac{18(251440)^2 + 7(403201)^2}{2.7.251440.403201}$$

and so on.

Ex. 2. $$\sqrt{\frac{346}{25}} = 3 + \frac{1}{\overset{*}{1}+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \frac{1}{\overset{*}{6}+}\ldots$$

Here the first *four* convergents are

$$\frac{3}{1},\ \frac{4}{1},\ \frac{11}{3},\ \frac{15}{4},$$

$$\therefore\ (\S\ 37)\ \text{the } \textit{seventh} \text{ convergent} = \frac{15 \times 4 + 11 \times 3}{3^2 + 4^2}$$

$$= \frac{93}{25}$$

$$\therefore\ (\S\ 38)\ \text{the } \textit{fourteenth} = \frac{346(25)^2 + 25(93)^2}{2.25.25.93}$$

$$= \frac{17299}{4650}$$

whence, in a similar manner, the *twenty-eighth;* and so on.

40. We have shown that a quadratic surd can always be expressed as an interminate continued fraction with unit-numerators and positive integral partial-denominators, that this continued fraction is a recurring one, that the cycle commences with the second partial quotient, that the last member of the cycle is double the first partial quotient of all, and that, if this last member be omitted, the cycle is the same when read backwards as when read

forwards. The converse of this, viz.: that any continued fraction of this nature must be equal to a quadratic surd, is established as a case of the following remarkable theorem.

41. *If* A, q_1, q_2 .... *be any finite quantities whatever, positive or negative, rational or irrational, &c., then*

$$A + \frac{1}{\underset{*}{q_1}+}\,\frac{1}{q_2+}\dots+\frac{1}{q_2+}\,\frac{1}{q_1+}\,\frac{1}{\underset{*}{2A}+}\dots = \sqrt{\frac{K(A,\, q_1\dots q_1,\, A)}{K(q_1\dots q_1)}}$$

For suppose $A + \frac{1}{\underset{*}{q_1}+}\,\frac{1}{q_2+}\dots+\frac{1}{q_2+}\,\frac{1}{q_1+}\,\frac{1}{\underset{*}{2A}+}\dots = x$

$$\text{then} \qquad A + \frac{1}{q_1+}\,\frac{1}{q_2+}\dots+\frac{1}{q_1+}\,\frac{1}{2A+x-A} = x.$$

$$\textit{i.e.} \quad \frac{K(A,\, q_1\dots q_1,\, A+x)}{K(q_1\dots q_1,\, A+x)} = x$$

$$\therefore \quad \frac{K(A,\, q_1\dots q_1,\, A) + xK(A,\, q_1\dots q_1)}{K(A,\, q_1\dots q_1) + xK(q_1\dots q_1)} = x$$

Clearing of fractions and simplifying, we have

$$K(A,\, q_1\dots q_1, A) = x^2K(q_1\dots q_1)$$

$$\therefore \quad x = \sqrt{\frac{K(A,\, q_1\dots q_1,\, A)}{K(q_1\dots q_1)}}$$

42. We thus see that if a continued fraction of the kind we have been considering be given we are able at once to find the quadratic surd equivalent to it.

EXAMPLES.

Ex. 1. Find the quadratic surds which are equal to

$$2 + \frac{1}{\underset{*}{1}+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{4}+}\dots \text{ and } \frac{1}{\underset{*}{3}+}\,\frac{1}{1+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{6}+}\dots$$

$$2 + \frac{1}{\underset{*}{1}+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{4}+}\dots = \sqrt{\frac{K(2,\, 1,\, 1,\, 1,\, 1,\, 2)}{K(1,\, 1,\, 1,\, 1)}}$$

$$= \sqrt{\frac{34}{5}}$$

$$\frac{1}{3+}\,\frac{1}{\underset{*}{1}+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{6}+}\dots = \text{reciprocal of } 3 + \frac{1}{\underset{*}{1}+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{\underset{*}{6}+}$$

$$= \sqrt{\frac{K(1,\ 5,\ 1)}{K(3,\ 1,\ 5,\ 1,\ 3)}}$$

$$= \sqrt{\frac{7}{104}}$$

Ex. 2. Find the general expression for all the quadratic surds which when expressed as continued fractions with unit-numerators have 1, 1, 1, 1, for the symmetric portion of the cycle of partial quotients.

$$n + \underset{*}{\frac{1}{1}} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \underset{*}{\frac{1}{2n}} + \ldots = \sqrt{\frac{K(n,\ 1,\ 1,\ 1,\ 1,\ n)}{K(1,\ 1,\ 1,\ 1)}}$$

$$= \sqrt{\frac{5n^2 + 6n + 2}{5}}$$

which is thus the general expression wanted. Putting $n = 2$ we have the particular example given above.

The wide generality of the theorem should not however, be forgotten.

EXAMPLE.

$$\sqrt[3]{2} + \underset{*}{\frac{1}{\sqrt[3]{-16}}} + \frac{1}{0} + \frac{1}{\sqrt[3]{-16}} + \underset{*}{\frac{1}{2\sqrt[3]{2}}} + \ldots$$

$$= \sqrt{\frac{K(\sqrt[3]{2},\ \sqrt[3]{-16},\ 0,\ \sqrt[3]{-16},\ \sqrt[3]{2})}{K(\sqrt[3]{-16},\ 0,\ \sqrt[3]{-16})}}$$

$$= \sqrt{\frac{8\sqrt[3]{-1} + 2\sqrt[3]{2}}{4\sqrt[3]{-2}}}$$

$$= \sqrt{(\sqrt[3]{4} + \tfrac{1}{2}\sqrt[3]{-1})}$$

Surds of the form $\sqrt{H}$, H being an integer.

43. Of course the preceding theorems all hold in reference to such quadratic surds as $\sqrt{17}$, $\sqrt{5}$, &c., where the extrahend (*i.e.*, the number under the root-sign) is an integer. It has only to be remembered that in this case $G = 1$, the *surd-dividend* is the same as the *given surd*, and the *highest integer in the surd-dividend* (I) is the same as the *unique partial quotient* (A), so that the expression of the theorems may be somewhat simplified. We now proceed to theorems referring to the special case only.

44. *When the extrahend is integral, a term equal to the unique partial quotient may occur in the cycle of partial quotients, but only in one position, viz.: as middle term.*

$$\text{For suppose} \qquad q_n = A$$
$$\text{then, (2)} \qquad A\,d_n = r_n + r_{n+1}$$
$$\therefore \quad (\S\ 10) \qquad d_n \text{ is not } > 2.$$

Now it is proved incidentally in § 21, that if $d_n = G$, then $q_n = 2\,A$, consequently d_n here cannot $= 1$ (*i.e.*, G) for (hyp.) $q_n = A$. Of necessity, then,

$$d_n = 2$$
$$\text{and} \ \therefore \ (\S\ 10) \qquad r_n = r_{n+1} = A$$

so that (§ 24) q_n must be the middle term of the cycle.

EXAMPLES.

$$\sqrt{31} = 5 + \underset{*}{\frac{1}{1+}}\ \frac{1}{1+}\ \frac{1}{3+}\ \frac{1}{5+}\ \frac{1}{3+}\ \frac{1}{1+}\ \frac{1}{1+}\ \underset{*}{\frac{1}{10+}} \ldots$$

$$\sqrt{131} = 11 + \underset{*}{\frac{1}{2+}}\ \frac{1}{4+}\ \frac{1}{11+}\ \frac{1}{4+}\ \frac{1}{2+}\ \underset{*}{\frac{1}{22+}} \ldots$$

$$\sqrt{(3n+1)^2 + 4n + 2} = (3n+1) + \underset{*}{\frac{1}{1+}}\ \frac{1}{2+}\ \frac{1}{3n+1+}\ \frac{1}{2+}\ \frac{1}{1+}\ \underset{*}{\frac{1}{2(3n+1)+}} \ldots$$

Here 5, 11, $3n+1$, the highest integers in the given surds are found to occur in the respective cycles of partial quotients, but only as middle terms.

45. *Although, when the extrahend is integral, the cycle of partial quotients may contain a term equal to the unique partial quotient* (§ 44), *and must contain one double of the same* (§ 21), *it cannot contain a term lying in value between these.*

$$\text{For (2)} \quad q_n d_n = r_n + r_{n+1} \ \overline{\overline{<}}\ 2\,A \qquad (\S\ 10)$$

so that if $q_n > A$, d_n must $= 1$, and consequently (§ 21) $q_n = 2\,A$. That is, every partial quotient greater than A must be equal to $2\,A$.

46. *When the extrahend is integral and there are* 2z *terms in the cycle of partial quotients,* $\mathrm{K}(\mathrm{A}\ldots \mathrm{q}_{z-2}) + \mathrm{K}(\mathrm{A}\ldots \mathrm{q}_z)$ *is exactly divisible by* $\mathrm{K}(\mathrm{q}_1\ldots \mathrm{q}_{z-1})$, *and further if the middle term be odd* $\mathrm{K}(\mathrm{A}\ldots \mathrm{q}_{z-1})$ *is exactly divisible by* $\mathrm{K}(\mathrm{q}_1\ldots \mathrm{q}_{z-2}) + \mathrm{K}(\mathrm{q}_1\ldots \mathrm{q}_z)$.

This is at once seen to follow from (27) § 32, if it be borne in mind that $K(q_1\ldots q_{z-1})$ is prime to $K(A\ldots q_{z-1})$ and, when q_z is odd, $K(q_1\ldots q_{z-2}) + K(q_1\ldots q_z)$ prime to $K(A\ldots q_{z-2}) + K(A\ldots q_z)$.

47. *When the extrahend* (H) *is a prime and there are* 2z *terms in the cycle of partial quotients, then*

$$\mathrm{K}(\mathrm{A}\ldots \mathrm{q}_{z-1}) = \mathrm{K}(\mathrm{q}_1\ldots \mathrm{q}_{z-2}) + \mathrm{K}(\mathrm{q}_1\ldots \mathrm{q}_z)$$
$$\mathrm{K}(\mathrm{A}\ldots \mathrm{q}_{z-2}) + \mathrm{K}(\mathrm{A}\ldots \mathrm{q}_z) = \mathrm{H}\,\mathrm{K}(\mathrm{q}_1\ldots \mathrm{q}_{z-1})$$

and the middle term is A *or* A − 1 *according as* A *is odd or even.*

The two identities and the fact that the middle term is odd may be demonstrated from (27) § 32. Supposing this done we have

$$K(A\ldots q_{z-1}) = K(q_1\ldots q_{z-2}) + K(q_1\ldots q_z)$$
$$\therefore \quad A\,K(q_1\ldots q_{z-1}) + K(q_2\ldots q_{z-1}) = 2\,K(q_1\ldots q_{z-2}) + q_z\,K(q_1\ldots q_{z-1})$$
$$\text{or} \quad (A - q_z)\,K(q_1\ldots q_{z-1}) + K(q_2\ldots q_{z-1}) = 2\,K(q_1\ldots q_{z-2})$$

Now, since $K(q_1\ldots q_{z-1}) > K(q_1\ldots q_{z-2})$, it follows that $A - q_z$ is not >1, that is, q_z is not $<A-1$; and we know (§ 45) that it is not $>A$, therefore, since it is odd, it must be A or $A-1$ according as A is odd or even.

EXAMPLES.

$$\sqrt{47} = 6 + \frac{1}{\underset{*}{1}} + \frac{1}{5} + \frac{1}{1} + \frac{1}{\underset{*}{12}} + \cdots$$

$$\sqrt{107} = 10 + \frac{1}{\underset{*}{2}} + \frac{1}{1} + \frac{1}{9} + \frac{1}{1} + \frac{1}{2} + \frac{1}{\underset{*}{20}} + \cdots$$

$$\sqrt{239} = 15 + \frac{1}{\underset{*}{2}} + \frac{1}{5} + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{15} + \frac{1}{4} + \frac{1}{2} + \frac{1}{1} + \frac{1}{5} + \frac{1}{2}$$
$$+ \frac{1}{\underset{*}{30}} + \cdots$$

48. *When the extrahend* (H) *is integral and the middle term* (q_z) *of the cycle of partial quotients is equal to the unique partial quotient, then*

$$K(q_2 \ldots q_{z-1}) = 2\, K(q_1 \ldots q_{z-2})$$

$$\textit{and}\ \ A^2 + \frac{2\, K(q_2 \ldots q_z)}{K(q_1 \ldots q_{z-1})} = H.$$

From (27) § 32 $\quad \dfrac{2\, K(A \ldots q_{z-1})}{K({}_1q \ldots q_z) + K(q_1 \ldots q_{z-2})} = \text{an integer},$

$$i.e. \quad \frac{2\, A\, K(q_1 \ldots q_{z-1}) + 2\, K(q_2 \ldots q_{z-1})}{q_z\, K(q_1 \ldots q_{z-1}) + 2\, K(q_1 \ldots q_{z-2})} = \ldots\ldots\ldots\ldots$$

But $q_z = A \qquad \therefore \quad K(q_2 \ldots q_{z-1}) = 2\, \mathrm{K}(q_1 \ldots q_{z-2})$

Further, we have at the same time,

$$K(A \ldots q_{z-1}) = K(q_1 \ldots q_z) + K(q_1 \ldots q_{z-2}) \qquad \text{and} \therefore\ (27)$$

$$\begin{aligned} H\, K(q_1 \ldots q_{z-1}) &= K(A \ldots q_{z-2}) + K(A \ldots q_z) \\ &= 2\, A\, K(q_1 \ldots q_{z-2}) + K(q_2 \ldots q_{z-2}) + A^2\, K(q_1 \ldots q_{z-1}) + K(q_2 \ldots q_z) \\ &= A\, K(q_2 \ldots q_{z-1}) + K(q_2 \ldots q_{z-2}) + A^2\, K(q_1 \ldots q_{z-1}) + K(q_2 \ldots q_z) \\ &= K(q_2 \ldots q_z) + A^2\, K(q_1 \ldots q_{z-1}) + K(q_2 \ldots q_z) \end{aligned}$$

$$\therefore\ H = A^2 + \frac{2\, K(q_2 \ldots q_z)}{K(q_1 \ldots q_{z-1})}.$$

49. By the second of the two identities just proved, there is suggested a theorem which may be derived from § 41, viz. :—

$$A + \underset{*}{\frac{1}{q_1}} + \ldots + \frac{1}{q_{z-1}} + \frac{1}{A} + \frac{1}{q_{z-1}} + \ldots + \frac{1}{q_1} + \underset{*}{\frac{1}{2\, A}} + \ldots$$

$$= \sqrt{\left\{ A + \frac{2\, K(q_2 \ldots q_z)}{K(q_1 \ldots q_{z-1}} \right\}}$$

50. Further, $K(q_2 \ldots q_{z-1})$ being even, it follows that $K(q_2 \ldots q_z)$ and $K(q_1 \ldots q_{z-1})$, and therefore also $K(q_2 \ldots q_z) \div K(q_1 \ldots q_{z-1})$ must be odd; and thus we observe, from the same identity, that when the square root of an integer has the middle term of its cycle of partial quotients equal to the unique partial quotient, the excess of the integer above the highest square in it is of the form $4\, s + 2$.

51. *Any integer* (H), *whose square root when expressed as a continued fraction with unit-numerators has an odd number of terms in its cycle of partial quotients, must be the sum of two integral squares prime to each other.*

Let the number of terms in the cycle be $2z+1$, then

$$d_z = d_{z+1}$$

and $\therefore$ (3) $\quad H - r^2_{z+1} = d^2_{z+1}$

so that $\quad H = d^2_{z+1} + r^2_{z+1}$

and we can show by a *reductio ad absurdum* that d_{z+1} and r_{z+1} are mutually prime.

EXAMPLES.

$$\sqrt{13} = 3 + \underset{*}{\frac{1}{1}} + \frac{1}{1} + \frac{1}{1} + \frac{1}{1} + \underset{*}{\frac{1}{6}} + \dots \qquad \text{and } 13 = 2^2 + 3^2$$

$$\sqrt{41} = 6 + \underset{*}{\frac{1}{2}} + \frac{1}{2} + \underset{*}{\frac{1}{12}} + \dots \qquad \text{and } 41 = 4^2 + 5^2$$

$$\sqrt{365} = 19 + \underset{*}{\frac{1}{9}} + \frac{1}{1} + \frac{1}{1} + \frac{1}{9} + \underset{*}{\frac{1}{38}} + \dots \qquad \text{and } 365 = 11^2 + 12^2$$

52. The converse of the preceding theorem is not universally true; thus $34 = 3^2 + 5^2$, while

$$\sqrt{34} = 5 + \underset{*}{\frac{1}{1}} + \frac{1}{4} + \frac{1}{1} + \underset{*}{\frac{1}{10}} + \dots$$

53. *The general expression for every integer whose square root, when expressed as a continued fraction with unit-numerators has* $q_1, q_2, \dots q_2, q_1$ *for the symmetric portion of its cycle of partial denominators, is*

$$\left\{\tfrac{1}{2}K(q_1..q_1)M - (-1)^l \tfrac{1}{2}K(q_1..q_2)K(q_2..q_2)\right\}^2 + K(q_1..q_2)M - (-1)^l K(q_2..q_2)^2$$

l *being the number of terms in the cycle.*

The general expression for *every* such number, fractional as well as integral, is (§ 41)

$$\frac{K(A, q_1, q_2 \dots q_2, q_1, A)}{K(q_1, q_2 \dots q_2, q_1)} \quad \dots\dots\dots \quad (a)$$

so that we have only to determine what form for A is necessary and sufficient to make this expression integral. Now

$$\begin{aligned} K(A, q_1 \dots q_1, A) &= AK(q_1 \dots q_1, A) + K(q_2 \dots q_1, A) \\ &= A^2 K(q_1 \dots q_1) + 2AK(q_1 \dots q_2) + K(q_2 \dots q_2) \end{aligned}$$

$$\therefore \quad \frac{K(A, q_1 \dots q_1, A)}{K(q_1 \dots q_1)} = A^2 + \frac{2AK(q_1 \dots q_2) + K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \dots (28)$$

so that it is necessary and sufficient that

$$\frac{2AK(q_1 \dots q_2) + K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \text{ be integral,}$$

$$\therefore \text{ that } \quad \frac{2AK(q_1 \dots q_2)^2 + K(q_1 \dots q_2)K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \quad \dots\dots\dots\dots\dots\dots ,$$

$$\textit{i.e.} \text{ that } \quad \frac{2A\{K(q_1 \dots q_1)K(q_2 \dots q_2) + (-1)^l\} + K(q_1 \dots q_2)K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \dots\dots ,$$

$$\therefore \text{ that } \quad \frac{(-1)^l 2A + K(q_1 \dots q_2)K(q_2 \dots q_2)}{K(q_1 \dots q_1)} \quad \dots\dots\dots\dots\dots\dots\dots\dots ,$$

or that $$A = \tfrac{1}{2}K(q_1 \dots q_1)\,\mathrm{M} - (-1)^l \tfrac{1}{2}K(q_1 \dots q_2)K(q_2 \dots q_2)$$

where M is zero or any integer.

Making this substitution for A in the alternative form of (a) given by (28) the second term becomes

$$\frac{K(q_1 \dots q_2)K(q_1 \dots q_1)\,\mathrm{M} - (-1)^l K(q_1 \dots q_2)^2 K(q_2 \dots q_2) + K(q_2 \dots q_2)}{K(q_1 \dots q_1)}$$

$$\text{or } K(q_1 \dots q_2)\,\mathrm{M} - (-1)^l \frac{K(q_2 \dots q_2)\ \{K(q_1 \dots q_2)^2 - (-1)^l\}}{K(q_1 \dots q_1)}$$

$$\text{or } K(q_1 \dots q_2)\,\mathrm{M} - (-1)^l K(q_2 \dots q_2)^2$$

and adding the other term we have the required expression

$$\{\tfrac{1}{2}K(q_1 .. q_1)\,\mathrm{M} - (-1)^l \tfrac{1}{2}K(q_1 .. q_2)K(q_2 .. q_2)\}^2 + K(q_1 .. q_2)\,\mathrm{M} - (-1)^l K(q_2 .. q_2)$$

54. Ex. Find all the integers whose square roots when expressed as continued fractions have each 1, 1, 2, 1, 1, for the symmetric portion of its cycle of partial denominators.

$$\begin{aligned} \text{Here } K(q_1 \ldots q_1) &= K(1, 1, 2, 1, 1) = 12 \\ K(q_1 \ldots q_2) &= K(1, 1, 2, 1) \quad = 7 \\ K(q_2 \ldots q_2) &= K(1, 2, 1) \qquad = 4 \end{aligned}$$

and therefore the general expression for the required integers is, by the preceding theorem,

$$(6\text{M} - 14)^2 + 7\text{M} - 16 .$$

Giving M the values 3, 4, 5, ... this becomes 21, 112, 275, ... and thus we know that

$$\sqrt{21} = 4 + \frac{1}{\overset{*}{1}+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\overset{*}{8}+}\cdots$$

$$\sqrt{112} = 10 + \frac{1}{\overset{*}{1}+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\overset{*}{20}+}\cdots$$

$$\sqrt{275} = 16 + \frac{1}{\overset{*}{1}+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\overset{*}{32}+}\cdots$$

and so on.

OTHER EXAMPLES.

$$\sqrt{(2\text{M}+1)^2+4} = (2\text{M}+1) + \frac{1}{\overset{*}{\text{M}}+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{\text{M}+}\,\frac{1}{\overset{*}{4\text{M}+2}+}\cdots$$

$$\sqrt{4\text{M}^2+4\text{M}-3} = 2\text{M} + \frac{1}{\overset{*}{1}+}\,\frac{1}{\text{M}-1+}\,\frac{1}{2+}\,\frac{1}{\text{M}-1+}\,\frac{1}{1+}\,\frac{1}{\overset{*}{4\text{M}}+}\cdots$$

55. *No integer can be found whose square root when expressed as a continued fraction with unit-numerators has* $q_1, q_2, \ldots q_2, q_1$ *for the symmetric portion of its cycle of partial denominators, unless either* $K(q_1 \ldots q_2)$ *or* $K(q_2 \ldots q_2)$ *be even.*

If both were odd we see that the general expression in § 53 would be fractional unless $K(q_1 \ldots q_1)$ were also odd, and this is impossible, for then

$$K(q_1 \ldots q_2) + K(q_2 \ldots q_2) \text{ which } = K(1, q_1 \ldots q_2)$$
$$\text{and } K(q_1 \ldots q_1) + K(q_1 \ldots q_2) \text{ which } = K(1, q_1 \ldots q_1)$$

would both be even, whereas they are seen to be mutually prime

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HISTORICAL NOTE.

THERE is one very important fact in the history of the subjec which has not yet found its way into text-books and books o reference—viz., that the square root of an integer was expressec as a continued fraction $\left(e.g. \quad \sqrt{18} = 4 + \frac{2}{8} {}_{+} \frac{2}{8} {}_{+} \frac{2}{8} {}_{+} \cdots\right)$ by Cataldi in his *Trattato del modo brevissimo di trovare la radice quadra delli numeri*, published in 1613, seven years before the birth o Lord Brounker, the commonly reputed introducer of continued fractions. (V. Libri, *Histoire des sciences Mathématiques en Italie* T. IV., p. 87-98.)